FIRST ORDER THEORY OF PERMUTATION GROUPS

BY

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ABSTRACT

We solve the problem of the elementary equivalence (definability) of the permutation groups over cardinals $\aleph_\alpha$. We show that it suffices to solve the problem of elementary equivalence (definability) for the ordinals α in certain second order logic, and this is reduced to the case of $\alpha < (2^{\aleph_0})^+$. We solve a problem of Mycielski and McKenzie on embedding of free groups in permutation groups, and discuss some weak second-order quantifiers.

0. Introduction

Let $\langle P_\alpha; \circ \rangle$ be the group of permutations of $\aleph_\alpha$, i.e., the set of ordinals $< \aleph_\alpha$ (which is isomorphic to the group of permutations of A if $|A| = \aleph_\alpha$). The question as to the elementary theories of permutation groups was raised by Fajtlowicz, and Isbell showed that those over uncountable sets and those over sets of cardinality $\leqq 2^{\aleph_0}$ can be characterized. The two specific problems are

1) when is $\langle P_\alpha; \circ \rangle \equiv \langle P_\beta; \circ \rangle$,?

2) when can $\langle P_\alpha; \circ \rangle$ be characterized by a sentence ψ (or set of sentences Γ) that is, $\langle P_\beta; \circ \rangle \vDash \psi$ iff $\beta = \alpha$. (We ignore for simplicity the permutation groups over finite sets.) McKenzie [9] shows that in $\langle P_\beta; \circ \rangle$ we can interpret $\langle \beta, < \rangle$ and derive from it some partial answers to questions (1) and (2). We give a necessary and sufficient condition for the elementary equivalence.

Our work was done independently of Pinus [12] who proved that we can interpret in $\langle P_\alpha; \circ \rangle$, $\langle \alpha, < \rangle$ with variables ranging over countable one-place functions and can derive more information on (1) and (2). We prove here that variables over relations of cardinality $\leqq$ continuum over $\langle \alpha, < \rangle$ can be interpreted (§2), and also that a "converse" is true (§3). Other connected works are Ershov

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[5] and [6], and Vazhenin and Rasin [16]. McKenzie [9] and [10] and Pinus [12] also contain more information.

In §4 we show that by Kino [8] we can reduce the general problem to the case $\alpha < (2^{\aleph_0})^+$ and in §5 we discuss some related problems and possible generalizations, and improve a result of McKenzie [10] on embedding free groups in permutation groups.

Let P_α^β be the family of permutations of $\aleph_\alpha$ which move $< \aleph_\beta$ elements. For example, for $\beta = \aleph_1$, De Bruijn [1,2] proves that the free group with $2^{\aleph_0}$ generators can be embedded (in P_α^1), McKenzie [10] shows that the free group with $\beth_3^+$ cannot be embedded, and we prove that the free group with $\beth_1^+ = (2^{\aleph_0})^+$ cannot be embedded. Theorem 5.1 gives the solution of the general problem.

In conclusion, we improve results and answers to particular questions of McKenzie [9] and Pinus [12].

Let $\Omega = (2^{\aleph_0})^+$, $|U_\alpha| = \min(2^{\aleph_0}, \aleph_\alpha)$. $\langle \alpha, U; < \rangle$ is the two-sorted model with domain α, U and the relation $<$ on α.

CONCLUSION 0.1. *$\langle P_\alpha; \circ \rangle \equiv \langle P_\beta; \circ \rangle$ iff the following conditions are satisfied where $\alpha = \Omega^\omega \alpha_\omega + \cdots + \Omega^n \alpha_n + \cdots + \alpha_0$, $\beta = \Omega^\omega \beta_\omega + \cdots + \Omega^n \beta_n + \cdots + \beta_0$, $\alpha_n, \beta_n < \Omega$*

1) *$\alpha < \Omega$ iff $\beta < \Omega$*
2) *if $\alpha < \Omega$, $\langle \alpha_0, U_{\alpha_0}; < \rangle \equiv_{L_2} \langle \beta_0, U_{\beta_0}; < \rangle$ (L_2 is second order logic)*
3) *if $\alpha \geqq \Omega$, $\langle \alpha_0, U^*; < \rangle \equiv_{L_2} \langle \beta_0, U^*; < \rangle$ ($|U^*| = 2^{\aleph_0}$)*
4) *for $0 < n < \omega$ $\langle \alpha_n, U^*; < \rangle \equiv_{L_2} \langle \beta_n, U^*; < \rangle$*
5) *$\mathrm{cf}(\Omega^\omega \alpha_\omega) \geqq \Omega$ iff $\mathrm{cf}(\Omega^\omega \beta_\omega) \geqq \Omega$*
6) *if $\mathrm{cf}(\Omega^\omega \alpha_\omega) < \Omega$ then $\langle \mathrm{cf}(\Omega^\omega \alpha_\omega), U^*; < \rangle \equiv_{L_2} \langle \mathrm{cf}(\Omega^\omega \beta_\omega), U^*; < \rangle$.*

PROOF. Immediate by Lemma 1.3, Conclusion 3.3, and Theorem 4.6.

CONCLUSION 0.2. *$\langle P_\alpha; \circ \rangle$ is definable by a sentence (set of sentences) iff (i) $\alpha = \Omega^n \alpha_n + \cdots + \Omega^1 \alpha_1 + \alpha_0$, $\alpha_i < \Omega$ and $\alpha \geqq \Omega$; $\langle \alpha_i, U^*; < \rangle$ are definable by a sentence (set of sentences) of L_2; or (ii) $\alpha < \Omega$ and $\langle \alpha_0, U_{\alpha_0}; < \rangle$ is definable by a sentence (set of sentences) of L_2.*

PROOF. By Lemma 1.3, Conclusion 3.3 and Theorem 4.6.

CONCLUSION 0.3.

a) *$\langle P_{\omega_1}; \circ \rangle$, $\langle P_\Omega; \circ \rangle$, $\langle P_{\Omega^n}; \circ \rangle$ ($n < \omega$) are definable by a sentence, and for no $\alpha \geqq \Omega^\omega$ is $\langle P_\alpha; \circ \rangle$ definable by a set of sentences.*

b) *If $\langle P_\alpha; \circ \rangle$, $\langle P_\beta; \circ \rangle$ are definable by a sentence then also $\langle P_{\alpha+\beta}; \circ \rangle$, $\langle P_{\alpha\beta}; \circ \rangle$ are definable, and if $\alpha, \beta < \Omega$, also $\langle P_{\alpha^\beta}; \circ \rangle$ is definable.*

c) *It is consistent with ZFC that there are* α, β *where* $2^{\aleph_\alpha} = \aleph_\beta$ *such that* $\langle P_\alpha; \circ \rangle$ *is definable by a sentence, but* $\langle P_\beta; \circ \rangle$ *is not definable even by a set of sentences.*

d) *The set of* $\aleph_\alpha$ *for which* $\langle P_\alpha; \circ \rangle$ *is definable by (a first-order) sentence, is not identical to the set of* α *for which* $\langle \aleph_\alpha; \rangle$ *is definable by a second order sentence.*

PROOF. By Conclusion 0.2.

We can consider our main results as determining the strength of the quantifier ranging over permutation. On possible quantifiers of this sort, see [14, 15] from which it follows that the permutational quantifier is very natural.

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1. Notation

By using multisorted models we can add a set of subsets, relations etc., as another sort of elements, and thus use first-order logic only. Cardinals are represented by λ, μ, κ; ordinals by $\alpha, \beta, \gamma, \delta, i, j, k$; and $\aleph_\alpha$ is the α-th cardinal. We identify α with $\{\beta : \beta < \alpha\}$, and $\aleph_\alpha$ with the first ordinal of that power. Let P_α be the set of permutations of $\aleph_\alpha$, E_α^κ the set of equivalence relations over $\aleph_\alpha$ with each equivalence class having a cardinality $< \kappa$ (if $\kappa > \aleph_\alpha$ we omit it), and $R_n^\kappa(A)[F_n^\kappa(A)]$ be the set of n-place relations (partial functions) with domain of cardinality $< \kappa$. The domain of an n-place relation r is $\bigcup \{\{x_1, \cdots, x_n\} : r(x_1, \cdots, x_n)\}$. A one-place relation is identified with the set it represents. $|A|$ is the cardinality of A.

Let $x, y, z \in \aleph_\alpha$, $f, g \in P_\alpha$, $e \in E_\alpha^{\aleph_1}$; $A, B \in R_1(\aleph_\alpha)$.

M and N are models. These are of the form

$$M_\alpha = \langle A_\alpha^1, A_\alpha^2, \cdots, A_\alpha^n;\ Q^1, \cdots, Q^m \rangle, \text{ where } Q^1, \cdots, Q^m$$

are relations and the A_α^n domains (e.g. α, $\aleph_\alpha$, $E_\alpha^\kappa, \cdots$). The equality between elements of the same sort and natural relations and operations will not be mentioned (e.g. $x e y$ for $e \in E_\alpha^\kappa$, $x, y \in \aleph_\alpha$). K^n denotes an indexed class $\{M_\alpha^n : \alpha \text{ an ordinal}\}$ of the same type; $L(K^n)$ is the corresponding first-order logic. The subsequent definitions can be naturally restricted to a subclass of ordinals (usually $\{\alpha : \aleph_\alpha \geqq 2^{\aleph_0}\}$).

DEFINITION 0.1. K^n can be interpreted in K^m (for $\alpha \in C$) if there is a recursive function $F: L(K^n) \to L(K^m)$ such that for any sentence $\psi \in L(K^n)$ and ordinal α, $(\alpha \in C)$

$$M^n_\alpha \models \psi \text{ iff } M^m_\alpha \models F(\psi).$$

DEFINITION 0.2. K^n can be explicitly interpreted in K^m if

$$M^n_\alpha = \langle A^1_\alpha, \cdots, A^k_\alpha; Q^1, \cdots Q^l \rangle, \; M^m_\alpha = \langle B^1_\alpha, \cdots, B^i_\alpha. \; R^1, \cdots, R^j \rangle,$$

and there are formulae $\phi_1(\bar{x}^1), \cdots, \phi_k(\bar{x}^k), \psi_1(\bar{x}^1, \bar{y}^1), \cdots, \psi_k(\bar{x}^k, \bar{y}^k)$, and $\theta_1, \cdots, \theta_l$ from $L(K^m)$ and functions $F^1_\alpha, \cdots, F^k_\alpha$ such that: for $1 \leqq \beta \leqq k$, F^β_α is a function from $\{\bar{a}: \bar{a} \text{ from } M^m_\alpha, M^m_\alpha \models \phi_\beta[\bar{a}]\}$ onto A^β_α, such that $F^\beta_\alpha[\bar{a}] = F^\beta_\alpha[\bar{b}]$ iff $M^m_\alpha \models \psi_\beta[\bar{a}, \bar{b}]$ and $M^n_\alpha \models Q^\gamma[\cdots, F_\alpha[\bar{a}], \cdots]$ iff $M^m_\alpha \models \theta_\gamma[\cdots, \bar{a}, \cdots]$ (all the sequences are of appropriate sorts).

LEMMA 1.1. *If K^n can be explicitly interpreted in K^m then K^n can be interpreted in K^m.*

LEMMA 1.2. *Interpretability and explicit interpretability are transitive and reflexive relations.*

LEMMA 1.3. *If K^n, K^m are bi-interpretable (i.e. each can be interpreted in the other) then*

a) $M^n_\alpha \equiv M^n_\beta$ *iff* $M^m_\alpha \equiv M^m_\beta$

b) M^n_α *is definable in K^n by a sentence (set of sentences) iff M^m_α is definable in K^m by a sentence (set of sentences).*

In defining interpretations, we shall be informal.

2. Interpretation in the permutation groups

We shall define indexed classes K^i and prove that K^{i+1} can be explicitly interpreted in K^i. In the next section we shall close the circle by interpreting K^1 in K^8, and thus get the desired result. Lemmas 2.1 to 2.3 were proved by McKenzie [9].

LEMMA 2.1. *K^2 can be explicitly interpreted in K^1 where*

$$M^1_\alpha = \langle P_\alpha; \circ \rangle, \; M^2_\alpha = \langle P_\alpha, \aleph_\alpha; \circ \rangle.$$

PROOF. (Hinted) The 2-cycles in P_α can be defined; therefore, an element of $\aleph_\alpha$ is defined by two 2-cycles.

LEMMA 2.2. *K^3 can be explicitly interpreted in K^2 where*

$$M^3_\alpha = \langle P_\alpha, \aleph_\alpha, R_1(\aleph_\alpha); \circ \rangle,$$

and there is a formula $\phi_{fin}(v) \in L(K^3)$ *defining the finite sets of* $R_1(\aleph_\alpha)$.

PROOF. When f ranges over P_α, $\{x : f(x) = x\}$ ranges over the subsets of $\aleph_\alpha$, except those whose complement has just one element. Therefore,

$$A_{f,g} = \{x : f(x) = x \vee g(x) = x\}$$

ranges over the subsets of $\aleph_\alpha$ and $x \in A_{f,g}$ can be expressed in $L(K^2)$. A set $A \in R_1(\aleph_\alpha)$ is finite iff there is no $f \in P_\alpha$ which maps it into a $B \subset A$, $B \neq A$.

LEMMA 2.3. K^4 *can be explicitly interpreted in* K^3 *where*

$$M_\alpha^4 = \langle P_\alpha, \aleph_\alpha, R_1(\aleph_\alpha), CR_\alpha; \circ, < \rangle,$$

CR_α *is the set of (finite and infinite) cardinals* $\leqq \aleph_\alpha$, $<$ *is the order on the cardinals, and* $\mathrm{cr}(A) = \lambda$ *is considered as one of the natural relations of* M_α^4, *where* $\mathrm{cr}(A)$ *is the cardinality of the set* A.

PROOF. We interpreted $\lambda \in CR_\alpha$ by $A \in R_1(\aleph_\alpha)$ of cardinality λ. Equality can be expressed in $L(K^3)$ as $\mathrm{cr}(A) = \mathrm{cr}(B)$ iff there is a permutation of $\aleph_\alpha$ mapping A onto B; or $\mathrm{cr}(A) = \mathrm{cr}(B) = \aleph_\alpha$, which is equivalent to the existence of $f, g \in P$ such that $A \cup \{f(x) : x \in A\} = B \cup \{g(x) : x \in A\} = \aleph_\alpha$. The order $\mathrm{cr}(A) < \mathrm{cr}(B)$ can be expressed by "$\mathrm{cr}(A) \neq \mathrm{cr}(B)$" and there is $f \in P_\alpha$ which maps A into B.

LEMMA 2.4. K^5 *can be explicitly interpreted in* K^4 *where*

$$M_\alpha^5 = \langle P_\alpha, \aleph_\alpha, R_1(\aleph_\alpha), CR_\alpha, E_\alpha^{\aleph_1}; \circ, < \rangle.$$

PROOF. Every permutation $f \in P_\alpha$ divides $\aleph_\alpha$ into its cycles, which are all of cardinality $\leqq \aleph_0$. More formally, for $f \in P_\alpha$, $e(f)$ is defined by: $xe(f)z$ iff for every $A \subseteq \aleph_\alpha$, $x \in A$, $(\forall y \in \aleph_\alpha)$ $[y \in A \leftrightarrow f(y) \in A]$ implies $z \in A$. Clearly $e(f) \in E_\alpha^{\aleph_1}$, and if $e \in E_\alpha^{\aleph_1}$, we define f_e as follows: for each e-equivalence class A, if A is finite let $A = \{a_1, \cdots, a_n\}$ and f_e is defined by $f_e(a_i) = a_{i+1}$ $(i = 1, \cdots, n-1)$, $f_e(a_n) = a_1$; if A is infinite let $A = \{a_n : n \text{ integer}\}$ and f_e is defined by $f_e(a_n) = a_{n+1}$. Clearly $e(f_e) = e$; therefore, when f ranges over P_α, $e(f)$ ranges over $E_\alpha^{\aleph_1}$, and $xe(f)y$ can be expressed in $L(K^4)$.

THEOREM 2.5. K^6 *can be explicitly interpreted in* K^5 *where*

$$M_\alpha^6 = \langle P_\alpha, \aleph_\alpha, R_1(\aleph_\alpha), CR_\alpha, E_\alpha^{\aleph_1}, \cdots, R_n^\Omega(CR_\alpha), \cdots; \circ < \rangle.$$

PROOF. For simplicity we shall interpret $R_2^\Omega(CR_\alpha)$ only. By pairing functions we can encode $R_n^\lambda(CR_\alpha)$ for $n > 2$. We shall prove that various notions can be expressed in $L(K^5)$. Let $[y]_e$ $(y \in \aleph_\alpha, e \in E_\alpha^{\aleph_1})$ be the e-equivalence class of y.

1) $x \in [y]_e =^{df} xey$.

Let $[y]_{e,f}$ be the model $\langle [y]_e; f' \rangle$, where $f \in P_\alpha$ and $f' = f \restriction \{z : zey \wedge f(z)ey\}$. We can express isomorphism between such models.

$$2) \qquad ([y_1]_{e_1,f_1} \cong [y_2]_{e_2,f_2}) \stackrel{df}{=} (\exists g)[(\forall x)[xe_1y_1 \leftrightarrow g(x)e_2y_2]$$
$$\wedge (\forall x)[xe_1y_1 \rightarrow (f_1(x)e_1y_1 \rightarrow f_2(g(x))e_2y_2)]$$
$$\wedge (\forall x)[xe_1y_1 \wedge f_1(x)e_1y_1 \rightarrow f_2(g(x)) = g(f_1(x))]].$$

This proof applies only for $\alpha > 0$, but we can correct this by quantifying over one-to-one unary functions instead of permutations, and these can be reduced to the sum of two permutations.

We can also express for fixed e, f, y, "the number of $[z]_{e,f}$ isomorphic to $[y]_{e,f}$ is λ".

$$3) \qquad [\mathrm{Pow}(y,e,f) = \lambda] \stackrel{df}{=} \quad (\exists A \in R_1(\aleph_\alpha))\, [(\forall x, z)$$
$$(x \in A \wedge z \in A \wedge x \neq z \rightarrow \neg\, xez) \wedge \mathrm{cr}(A) = \lambda \wedge (\forall x)\, (x \in A \rightarrow [x]_{e,f} \cong [y]_{e,f})$$
$$\wedge (\forall x)\, ([x]_{e,f} \cong [y]_{e,f} \rightarrow (\exists z)\, (z \in A \wedge zex))].$$

Now define a 2-place relation $r = r(e,f,A; e_1,f_1,A_1; g)$ over CR_α as follows: $r(\lambda,\mu)$ holds iff there are $x, y \in A$ such that $\mathrm{Pow}(x,e,f) = \lambda$, $\mathrm{Pow}(y,e,f) = \mu$, and there is $z \in A_1$, such that $[z]_{e_1,f_1} \cong [x]_{e,f}$ and $[g(z)]_{e_1,f_1} \cong [y]_{e,f}$. Clearly this can be expressed in $L(K^5)$.

$$4) \quad r(e,f,A;e_1,f_1,A_1;g)\,[\lambda,\mu] \stackrel{df}{=} (\exists xyz)\, (\mathrm{Pow}(x,e,f) = \lambda \wedge x \in A \wedge y \in A$$
$$\wedge \mathrm{Pow}(y,e,f) = \mu \wedge z \in A_1 \wedge [z]_{e_1,f_1} \cong [x]_{e,f} \wedge [g(z)]_{e_1,f_1} \cong [y]_{e,f}).$$

To finish the proof we need to prove only that for any $r \in R_2^2(CR_\alpha)$ we can find e,f,A,e_1,f_1,A_1,g such that $r = r(e,f,A;e_1,f_1,A_1;g)$. Let B be the domain of r so $|B| \leqq 2^{\aleph_0}$; $|B| \leqq |\alpha| + \aleph_0 \leqq \aleph_\alpha$, and $B = \{\lambda_i : i < i_0 \leqq 2^{\aleph_0}\}$. For each $i \leqq i_0$ choose a model $\langle A_i^0; f_i^0 \rangle$ where f_i is a permutation of A_i^0, $|A_i^0| = \aleph_0$; and for $i \neq j$, $\langle A_i^0; f_i^0 \rangle \not\cong \langle A_j^0; f_j^0 \rangle$ (this is possible because for each set I of natural numbers $n > 0$ there is such a model $\langle A; f \rangle$ which has an n-cycle iff $n \in I$; an n-cycle is $\{x_1, \cdots, x_n\} \subseteq A$, the x_i distinct and $f(x_i) = x_{i+1}$, $f(x_n) = x_1$). As $\sum_{i<i_0} \lambda_i \leqq \aleph_\alpha \aleph_\alpha = \aleph_\alpha$ and $\aleph_0 \aleph_\alpha = \aleph_\alpha$, we can easily find $e \in E_\alpha^{\aleph_1}$ and $f \in P_\alpha$ such that for $i < i_0$ $|\{[x]_{e,f} : x \in \aleph_\alpha, [x]_{e,f} \cong \langle A_i^0; f_i \rangle\}| = \lambda_i$ (the other e-equivalence classes may be chosen isomorphic to $\langle A_{i_0}^0; f_{i_0}^0 \rangle$). For each $i < i_0$ choose $x_i \in \aleph_\alpha$, such that $[x_i]_{e,f} \cong \langle A_i^0; f_i^0 \rangle$ and let $A = \{x_i : i < i_0\}$. For each pair $\langle \lambda, \mu \rangle$ such that $r(\lambda,\mu)$ holds, choose two disjoint countable subsets of $\aleph_\alpha$; $C^1_{\langle\lambda,\mu\rangle}$, and $C^2_{\langle\lambda,\mu\rangle}$ and $z_{\langle\lambda,\mu\rangle} \in C^1_{\langle\lambda,\mu\rangle}$; and choose them so that the C's are disjoint also for different pairs. Now define f_1 so that when $r(\lambda_i, \lambda_j)$

$$\langle C^1_{\langle\lambda_i,\lambda_j\rangle}; f_1 \restriction C^1_{\langle\lambda_i,\lambda_j\rangle}\rangle \cong \langle A^0_i; f^0_i\rangle,$$

$$\langle C^2_{\langle\lambda_i,\lambda_j\rangle}; f_1 \restriction C^2_{\langle\lambda_i,\lambda_j\rangle}\rangle \cong \langle A^0_j; f^0_j\rangle,$$

and $A_1 = \{z_{\langle\lambda,\mu\rangle}: r(\lambda,\mu)\}$, and let g be such that $g(z_{\langle\lambda,\mu\rangle}) \in C^2_{\langle\lambda,\mu\rangle}$. It is easy to check that $r = r(e,f,A; e_1, f_1, A_1; g)$, where e_1 is chosen accordingly.

LEMMA 2.6. *K^7 can be explicitly interpreted in K^6 where*

$$M^7_\alpha = \langle \alpha, U_\alpha, \cdots, R^\Omega_n(\alpha \cup U_\alpha), \cdots, ; <\rangle$$

where U_α is any set disjoint from α of cardinality $\min(\aleph_\alpha, 2^{\aleph_0})$ *and $<$ is the natural order of ordinals.*

PROOF. We interpret the element β of α by $\aleph_\beta$. All we need to prove is that $\{\lambda: \aleph_0 \leqq \lambda < \aleph_\alpha\}$ is definable in M^6_α. This is true because $\aleph_\alpha \in R_1(\aleph_\alpha)$ is definable, hence $\lambda < \mathrm{cr}(\aleph_\alpha)$ is definable; and by Lemma 2.2 the finite sets are definable, hence also the finite and infinite cardinals. Interpret $u \in U_\alpha$ as isomorphism types of $[x]_{e,f}$ when $\aleph_\alpha \geqq 2^{\aleph_0}$, and as elements of $\aleph_\alpha$ otherwise. We leave $R^\Omega_n(\alpha \cup U_\alpha)$ to the reader.

LEMMA 2.7. *K^8 can be interpreted explicitly in K^7 where*

$$M^8_\alpha = \langle CR_\alpha, U_\alpha, \cdots, R^\Omega_n(CR_\alpha \cup U_\alpha), \cdots, \cdots, F^\Omega_n(CR_\alpha \cup U_\alpha), \cdots, P(U_\alpha); <, \Sigma\rangle$$

where for $F: U_\alpha \to CR_\alpha$, $\Sigma(F) = \Sigma_{u\in U_\alpha} F(u)$, and $P(U_\alpha)$ is the set of permutations of U_α.

PROOF. Interpret $\aleph_\beta$, $\beta < \alpha$ by β; interpret $\lambda < \aleph_0$ by the subsets of U_α of cardinality λ; and interpret $\aleph_\alpha$ by U_α.

3. Interpreting the permutation groups

THEOREM 3.1. *K^1 can be interpreted in K^8.*

PROOF. For a sequence $\bar{f} = \langle f_1, \cdots, f_n\rangle$ of permutations (of a set A^*_0) define the equivalence relation eq $(\bar{f})$ as follows: an eq$(\bar{f})$-equivalence class A is a minimal set such that $x \in A \leftrightarrow f_i(x) \in A$ for any $1 \leqq i \leqq n$. Let the eq$(\bar{f})$-equivalence class of x be $A(x,\bar{f})$, and $N(x,\bar{f}) = \langle A(x,\bar{f}); \cdots, f_i \restriction A(x,\bar{f}), \cdots\rangle$. Clearly each eq$(\bar{f})$-equivalence class has cardinality $\leqq \aleph_0$. The characteristic function ch $=$ ch$[\bar{f}]$ of

$\bar{f}$, gives for any model $N = (A, f_1^0, \cdots, f_n^0)$ (where f_i^0 is a permutation of A, and $\mathrm{eq}(f_1^0, \cdots, f_n^0)$ has one equivalence class) the cardinality of $\{N(x, \bar{f}) : x \in A_0^*, N(x, \bar{f}) \cong N\}$.

A representation $\langle \bar{f}^*, F \rangle$ of a sequence $\bar{f} = \langle f_1, \cdots, f_n \rangle$, $f_i \in P_\alpha$ consists of $\bar{f}^* = \langle f_1^*, \cdots, f_n^* \rangle$, where f_i^* is a permutation of U_α, and F is a function from U_α into CR_α such that $\mathrm{ch}[\bar{f}^*]$ has the values $\mathrm{cr}(U_\alpha)$ or 0 and for $u \in U$, $F(u) = \mathrm{ch}[\bar{f}](N(u, \bar{f}^*))$ and for each $x \in \aleph_\alpha$, $\mathrm{ch}\,[\bar{f}^*](N(x, \bar{f})) > 0$. Clearly each $\bar{f}$ has a representation. Notice that if $\bar{f}^1$, $\bar{f}^2$ have a common representation, then $\langle \aleph_\alpha; \cdots, f_i^1, \cdots \rangle \cong \langle \aleph_\alpha; \cdots f_i^2, \cdots \rangle$. It suffices to prove:

LEMMA 3.2. *For each formula $\phi \in L(K^1)$, $\phi = \phi\,(v_1, \cdots, v_n)$ (that is $v_1, \cdots, v_n$ include all its free variables) we can define inductively (in a uniform way) a formula $\psi \in L(K^8)$, $\psi = \psi(v_1, \cdots, v_n, v)$, v_i $(i = 1, \cdots, n)$ range over $P(U_\alpha)$, and v ranges over functions from U_α to CR_α, such that if $f_1, \cdots, f_n \in P_\alpha$,* and *$\langle \bar{f}^*, F \rangle$ is any representation of $\bar{f} = \langle f_1, \cdots, f_n \rangle$, then $M_\alpha^1 \vDash \phi[f_1, \cdots, f_n]$ iff $M_\alpha^8 \vDash \psi[f_1^*, \cdots, f_n^*, F]$.*

PROOF OF THE LEMMA. There is a formula $\phi_0 \in L(K^8)$ such that $M_\alpha^8 \vDash \phi_0^n[\bar{f}^*, F]$ iff $\langle \bar{f}^*, F \rangle$ is a representation of some $\bar{f}$. This formula says that $\alpha > 0 \to \mathrm{eq}(\bar{f})$ has $\aleph_\alpha$ equivalence classes, i.e.,

$$(\exists F^1)\,(\exists A \subseteq U_\alpha)\,[(\forall x \in U_\alpha)\,(\exists !\, y \in A)\,(N(x, \bar{f}^*) \cong N(y, \bar{f}^*)) \wedge \Sigma F^1 = \aleph_\alpha$$
$$\wedge\ \forall x (x \in A \to F^1(x) = F(x)) \wedge (\forall x \in U_\alpha)\,(x \notin A \to F^1(x) = 0)];$$

and that each $\mathrm{eq}(\bar{f}^*)$-equivalence class is isomorphic to $|U_\alpha|$ others, F is a function from U_α into CR_α, and $F(u)$ depends only on the isomorphism type of $N(u, \bar{f}^*)$.

REMARKS.

1) Clearly $\aleph_0$ is definable here.

2) We should say more for the case where $\alpha = 0$.

There is a formula $\phi_1^n \in L(K^8)$ such that if $\bar{f}, \bar{g}$ are sequences of length n from P_α, and $\langle \bar{f}^*, F \rangle$, $(\bar{g}^*, G\rangle$ are the corresponding representations then $M_\alpha^8 \vDash \phi_1^n[\bar{f}^*, F, g^*, G]$ iff $\langle \aleph_\alpha, \bar{f} \rangle \cong \langle \aleph_\alpha, \bar{g} \rangle$. ϕ_1^n says that $N(u_1, \bar{f}^*) \cong N(u_2, \bar{g}^*)$ implies $F(u_1) = G(u_2)$, and $(\forall u_1)$ $[F(u_1) > 0 \to (\exists u_2)$ $(N(u_1, \bar{f}^*) \cong N(u_2, \bar{g}^*))]$ and $(\forall u_2)\,[G(u_2) > 0 \to (\exists u_1)\ N(u_2, \bar{g}^*) \cong N(u_1, \bar{f}^*)]$.

Also, there is a formula $\phi_2^n \in L(K^8)$ such that if $\bar{f} = \langle f_1, \cdots, f_n \rangle$, where $f_i \in P_\alpha$, $\langle f^*, F \rangle$ is a representation of $\bar{f}$, and $\bar{g} = \langle f_1, \cdots, f_{n-1} \rangle$, then for any G, $M_\alpha^8 \vDash \phi_2^n[\bar{f}^*, F, G]$ iff $\langle \bar{g}^*, G \rangle$ represents $\bar{g}$, where $\bar{g}^* = \langle f_1^*, \cdots, f_{n-1}^* \rangle$. Hence, ϕ_2^n defines G uniquely by f^*, F. ϕ_2^n says for $u \in U_\alpha$ $G(u) = \Sigma\,\{F(u_1) : u_1 \in A\}$

where $A \subseteq U_\alpha$ and for each $u_1 \in U_\alpha$, if $N(u_1, \bar{g}^*) \cong N(u, \bar{g}^*)$ *then* there is a unique $u_2 \in A$ for which $N(u_2, \bar{f}^*) \cong N(u_1, \bar{f}^*)$.

Continuing the proof of the lemma, we notice that ψ depends on ϕ and on $\{v_1, \cdots, v_n\}$. We prove the existence of ψ by induction on ϕ simultaneously for any suitable $\{v_1, \cdots, v_n\}$.

If ϕ is atomic, that is $[v_i = v_j]$ or $[v_i o v_j = v_k]$, then $\psi = (\forall x \in U_\alpha)\ [v(x) > 0 \rightarrow v_i(x) = v_j(x)]$ or $\psi = (\forall x \in U_\alpha)\ [v(x) > 0 \rightarrow v_i(v_j(x)) = v_k(x)]$ will do (where the v's now range over $P(U_\alpha)$.

If for ϕ, $\{v_1, \cdots, v_n\}$ we choose ψ, then for $\neg\, \phi$, $\{v_1, \cdots, v_n\}$ we shall choose $\neg\, \psi$.

If $\{v_1, \cdots, v_n\}$ includes the free variables of $\phi_1 \wedge \phi_2$, and for ϕ_i, $\{v_1, \cdots, v_n\}$ we choose ψ_i $(i = 1,2)$, then for $\phi_1 \wedge \phi_2$ we choose $\psi_1 \wedge \psi_2$.

If $\phi^* = (\exists v_n)\ \phi(v_1, \cdots, v_n)$, $\{v_1, \cdots, v_{n-1}\}$ includes the free variables of ϕ^*, and for ϕ, $\{v_1, \cdots, v_n\}$ we have chosen ψ, then for ϕ^* we choose

$$\psi^* = \psi^*(v_1, \cdots, v_{n-1}, v) = (\exists v_1^1, \cdots, v_n^1 v^1 v^2)\ [\phi_0^n(v_1^1, \cdots, v_n^1, v^1)$$

$$\wedge\ \phi_1^n(v_1, \cdots, v_{n-1}, v;\ v_1^1, \cdots, v_{n-1}^1, v^2) \wedge \phi_2^n(v_1^1, \cdots, v_n^1, v^1, v^2) \wedge \psi(v_1^1, \cdots, v_n^1, v^1)].$$

Clearly it is suitable.

CONCLUSION 3.3. *Any two of K^i, $i = 1, \cdots, 8$ are bi-interpretable. In particular the permutation groups $\langle P_\alpha; \circ \rangle$ and the $\langle \alpha, U_\alpha; < \rangle$ in the logic $L_2(\Omega)$ are bi-interpretable. (For a definition of $L_2(\Omega)$, see below.)*

PROOF. By 2.1–2.7 and 3.1, remembering 1.1, 1.2.

4. The $L_2(\Omega)$ theories of ordinals

$L_2(\Omega)$ is the second order logic where the higher type variables range over relations (functions) with domain of power $< \Omega$.

Note the following lemma (Feferman and Vaught [7]):

LEMMA 4.1.

A) *If $\gamma_i = \sum_{j<\beta} \alpha_j^i$, $i = 0, 1$ and*

$$\langle \alpha_j^0, U_{\alpha_j^0}; < \rangle \equiv_{L_2(\Omega)} \langle \alpha_j^1, U_{\alpha_j^1}; < \rangle$$

for every $j < \beta$, then $\langle \gamma_0, U_{\gamma_0}; < \rangle \equiv_{L_2(\Omega)} \langle \gamma_1, U_{\gamma_1}; < \rangle$.

B) *For every $n < \omega$ we can replace the full $L_2(\Omega)$ by the set of $\psi \in L_2(\Omega)$ with quantifier depth $[\mathrm{df}(\psi)] \leqq n$.*

REMARK.

1) Elementary equivalence for this set will be denoted by $\equiv^n_{L_2(\Omega)}$.

2) Let us define $\mathrm{df}(\psi)$. When ψ is an atomic formula, $\mathrm{df}(\psi)$ is 0; when $\psi = \neg\phi$, $\mathrm{df}(\psi)$ is $\mathrm{df}(\phi)$; when $\psi = \phi_1 \wedge \phi_2$, $\mathrm{df}(\psi)$ is $\max\{\mathrm{df}(\phi_1), \mathrm{df}(\phi_2)\}$; and when $\psi = (\exists x)\phi$, $\mathrm{df}(\psi)$ is $1 + \mathrm{df}(\phi)$.

LEMMA 4.2. a) *For* $\alpha \geqq 2^{\aleph_0}$, K^7 *and* K^9 *are bi-interpretable explicitly where* $M^9_\alpha = \langle \alpha, \cdots, R^\Omega_n(\alpha) \cdots; < \rangle$ *(this is the same as* $L_2(\Omega)$ *on* $\langle \alpha, < \rangle$*).*

b) *For* $\alpha < \Omega$ $[= (2^{\aleph_0})^+]$, $L_2(\Omega)$ *is the same as second order logic, that is* $M^9_\alpha = \langle \alpha, \cdots, R_n(\alpha), \cdots; < \rangle$.

It is clear that every sentence ψ in $L_2(\Omega)$ is equivalent to a sentence ψ^* in $L_{\Omega,\Omega}$ of finite depth $\mathrm{df}(\psi)$ ($L_{\Omega,\Omega}$ is the infinitary logic with conjunctions over continuum many formulae, and quantification ($\exists$ or $\forall$) over strings of $\leqq 2^{\aleph_0}$ variables). From Kino [8] it is clear that if the ordinal α has cofinality $\geqq \Omega$, and is divisible by $\Omega^{df(\psi)}$then $\langle \alpha; < \rangle \vDash \psi^*$ iff $\langle \mathrm{Or}; < \rangle \vDash \psi^*$ (where Or is the class of ordinals).

If α, β have cofinality $\geqq \Omega$ and are divisible by $\Omega^{df(\psi)}$ then $\langle \alpha; < \rangle \vDash \psi^*$ iff $\langle \beta; < \rangle \vDash \psi^*$. Hence if $\alpha, \beta > 0$ are divisible by Ω^ω and have cofinality $\geqq \Omega$ then this holds for any ψ^* ($\psi \in L_2(\Omega)$); therefore $\langle \alpha; < \rangle \equiv_{L_2(\Omega)} \langle \beta; < \rangle$. If α, β are divisible by Ω^ω, and have cofinality $\kappa < \Omega$, then for any n, $\alpha = \sum_{i<\kappa} \alpha_i$, $\beta = \sum_{i<\kappa} \beta_i$, and α_i, β_i have cofinality $\geqq \Omega$ and are divisible Ω^n; hence, $\langle \alpha_i; < \rangle \equiv^n_{L_2(\Omega)} \langle \beta_i; < \rangle$. Hence by Lemma 4.1 (B), $\langle \alpha; < \rangle \equiv^n_{L_2(\Omega)} \langle \beta, < \rangle$. As this holds for any n, $\langle \alpha; < \rangle \equiv_{L_2(\Omega)} \langle \beta; < \rangle$.

This discussion proves

LEMMA 4.3. *If* $\alpha, \beta > 0$ *are divisible by* Ω^ω, *and their cofinalities are equal or* $\geqq \Omega$ *then* $\langle \alpha, < \rangle \equiv_{L_2(\Omega)} \langle \beta; < \rangle$, *or equivalently* $\langle \alpha, U_\alpha; < \rangle \equiv_{L_2(\Omega)} \langle \beta, U_\beta; < \rangle$ *(necessarily* $\alpha, \beta \geqq \Omega$*).*

Let U^* be any set of cardinality $2^{\aleph_0}$ so for $\aleph_\alpha \geqq 2^{\aleph_0}$, without loss of generality, $U_\alpha = U^*$. We would like to weaken the demand on the equality of cofinalities. We can easily generalize Lemma 4.1 to multiplication.

LEMMA 4.4. *For any* $n < \omega$ *there is* m *such that if* $\alpha_i = \beta_i \gamma_i$, $i = 1,2$, $\langle \beta_1, U^*; < \rangle \equiv^m_{L_2(\Omega)} \langle \beta_2, U^*; < \rangle$, *and* $\langle \gamma_1, U^*; < \rangle \equiv^m_{L_2(\Omega)} \langle \gamma_2, U^*; < \rangle$ *then* $\langle \alpha_1, U^*; < \rangle \equiv^n_{L_2(\Omega)} \langle \alpha_2, U^*; < \rangle$.

From the above follows

LEMMA 4.5. *If* $\alpha, \beta > 0$ *are divisible by* Ω^ω, *and* $\mathrm{cf}(\alpha)$, $\mathrm{cf}(\beta) \geqq \Omega$ *or*

$\langle \mathrm{cf}(\alpha), U^*; <\rangle \equiv_{L_2(\Omega)} \langle \mathrm{cf}(\beta), U^*; <\rangle$ *then* $\langle \alpha, U_\alpha; <\rangle \equiv_{L_2(\Omega)} \langle \beta, U_\beta; <\rangle$ $(U_\alpha = U_\beta = U^*)$.

THEOREM 4.6. *For any ordinals* α, β *we have a unique representation* $\alpha = \Omega^\omega \alpha_\omega + \cdots + \Omega^n \alpha_n + \cdots + \Omega^1 \alpha_1 + \alpha_0, \alpha_n < \Omega$ *for* $n < \omega$, *and only finitely many* α_n *are* $\neq 0$; $\beta = \Omega^\omega \beta_\omega + \cdots + \Omega^n \beta_n + \cdots + \Omega^1 \beta_1 + \beta_0$, $\beta_n < \Omega$ *for* $n < \omega$ *and only finitely many* β_n *are* $\neq 0$.

Now $\langle \alpha, U_\alpha; <\rangle \equiv_{L_2(\Omega)} \langle \beta, U_\beta; <\rangle$ *iff the following conditions are satisfied*:

1) $\alpha < \Omega$ *iff* $\beta < \Omega$
2) *if* $\alpha < \Omega$, $\langle \alpha_0, U_{\alpha_0}; <\rangle \equiv_{L_2} \langle \beta_0, U_{\beta_0}; <\rangle$
3) *if* $\alpha \geqq \Omega$, $\langle \alpha_0, U^*; <\rangle \equiv_{L_2} (\beta_0, U^*; <\rangle$
4) *for* $0 < n < \omega$ $\langle \alpha_n, U^*; <\rangle \equiv_{L_2} \langle \beta_n, U^*; <\rangle$
5) $\mathrm{cf}(\Omega^\omega \alpha_\omega) \geqq \Omega$ *iff* $\mathrm{cf}(\Omega^\omega \beta_\omega) \geqq \Omega$
6) *if* $cf(\Omega^\omega \alpha_\omega) < \Omega$ *then* $\langle \mathrm{cf}(\Omega^\omega \alpha_\omega), U^*; <\rangle \equiv_{L_2} \langle \mathrm{cf}(\Omega^\omega \beta_\omega), U^*; <\rangle$.

We have proven the sufficiency of the conditions. Their necessity is easy to prove, e.g., $\alpha < \Omega$ iff $\langle \alpha; <\rangle \models (\exists A)\ (\forall x)\ (x \in A)$.

5. Discussion

a) Clearly we can interpret in the group of permutations of $\aleph_\alpha$: (1) one-to-one functions from $\aleph_\alpha$ into $\aleph_\alpha$, (2) equivalence relations with $\leqq 2^{\aleph_0}$ equivalence class, (3) the lattice of $E_\alpha^{\aleph_1}$ and (4) $E_\alpha^{\aleph_1}$ partially ordered. Except for (2) also the converses are true.

b) Let P_α^β be the group of permutations $f \in P_\alpha$, $|\{x : f(x) \neq x\}| < \aleph_\beta$. It is easy to see by Vaught's test that if $\beta \leqq \alpha \leqq \gamma$ then $\langle P_\alpha^\beta, \circ\rangle$ is an elementary submodel of $\langle P_\gamma^\beta; \circ\rangle$, and we can, with no difficulty, describe the elementary theories of $\langle P_\alpha^\beta; \circ\rangle$ in a way parallel to the description for $\langle P_\alpha, \circ\rangle$.

McKenzie [10], solving the question of Mycielski [11], asks when $FG(\lambda)$ (the free group with λ-generators) can be embedded in P_α^β. By De Bruijn [1, Th. 4.2], if there is $\gamma < \beta$ such that $2^{\aleph_\gamma} \geqq \lambda$, and $\gamma \leqq \alpha$ then there is such an embedding. McKenzie [10] shows that if λ is big enough relative to $\aleph_\alpha$, then this cannot be done.

Let $\{x_i : i < \lambda\}$ be the generators of $FG(\lambda)$, and F an embedding of $FG(\lambda)$ into P_α^β. McKenzie ([10] p. 57) shows, using a partition relation $\mu \to ((2^\kappa)^+)_2^{3\kappa}$ from Erdös, Hajnal and Rado [3], that for λ big enough there is $I \subseteq \lambda$, $|I| > \sum_{\kappa < \aleph_\beta} 2^\kappa$ such that

(*) for $i_1 < j_1 < k_1 \in I$, $i_2 < j_2 < k_2 \in I$, there is a permutation of $\aleph_\alpha$ which takes $\langle F(x_{i_1}), F(x_{j_1}), F(x_{k_1})\rangle$ to $\langle F(x_{i_2}), F(x_{j_2}), F(x_{k_2})\rangle$

and from this he gets a contradiction. Now if $\lambda > \sum_{\kappa<\aleph_\beta} 2^\kappa$, let

$$A_i = \{y : y \in \aleph_\alpha,\ F(x_i)[y] \neq y\}\ \ (F(x_i) \in P_\alpha^\beta);$$

therefore $|A_i| < \aleph_\beta$. Hence, by Erdös and Rado [4] there is $I \subseteq \lambda$, $|I| > \sum_{\kappa<\aleph_\beta} 2^\kappa$ such that for $i \neq j \in I$, $A_i \cap A_j = A$ (i.e., any two A_i have the same intersection); therefore, $|A| < \aleph_\beta$. Hence there is $J \leqq I$, $|J| > \sum_{\kappa<\aleph_\beta} 2^\kappa$ such that for $i \neq j \in J$, $\langle A_j; F(x_j), a\rangle_{a \in A} \cong \langle A_i; F(x_i), a\rangle_{a\in A}$. Clearly (*) is satisfied, which McKenzie shows is impossible. Therefore, if $FG(\lambda)$ can be embedded in P_α^β then $\lambda \leqq \sum_{\kappa<\aleph_\beta} 2^\kappa$, and

$$\lambda \leqq \sum\{2^\kappa : \kappa \leqq \aleph_\alpha,\ \kappa < \aleph_\beta\}.$$

The remaining problem is for β a limit ordinal, $\beta \leqq \alpha$, $\lambda = \sum_{\gamma<\beta} 2^{\aleph_\gamma}$ but $\gamma < \beta \rightarrow 2^{\aleph_\gamma} < \lambda$. Let $g_k = F(x_k) \circ F(x_0) \circ F(x_k)^{-1} \circ F(x_0)^{-1}$. Then $\{g_\delta : 0 < \delta < \lambda,\ \delta$ a limit ordinal$\}$ generates a free subgroup of cardinality λ, and $|\{x : g_\delta(x) \neq x\}| \leqq |\{y : F(x_0)(y) \neq y\}| + \aleph_0 \leqq \aleph_\gamma < \aleph_\beta$, so we get a contradiction as before.

THEOREM 5.1. *The free group of cardinality λ is isomorphic to a subgroup of P_α^β* (i.e., *the group of permutations of $\aleph_\alpha$ which moves $< \aleph_\beta$ elements*) *iff for some $\aleph_\gamma < \aleph_\beta$ $2^{\aleph_\gamma} \geqq \lambda$.*

c) In the same way we prove Conclusion 3.3, we can prove

THEOREM 5.2. *K^{10}, K^{11} are bi-interpretable where*

1) $M_\alpha^{10} = \langle \aleph_\alpha, E_\alpha^\kappa;\rangle$

2) $M_\alpha^{11} = \langle \alpha, U_\alpha, \cdots, R_n(\alpha \cup U_\alpha), \cdots; <\rangle$ *where $<$ is the order of ordinals and* $|U_\alpha| = \min(\aleph_\alpha, (\sum_{\mu<\kappa} 2^\mu) + 2^{\aleph_0})$. *($\kappa$ is any regular cardinal; the interpretation is independent of κ).*

The essential property of E_α^κ we used is that for any $e \in E_\alpha^\kappa$, $\langle \aleph_\alpha; e\rangle$ is the direct sum of models, each of cardinality $< \kappa$. Thus, there are many variants of our theorems.

As an easy corollary of Theorem 5.2 we have a well-known theorem of Rabin [17] that allowing quantifications over arbitrary equivalence relations is equivalent to full second order logic. (In fact, more is proved there).

We can seek another generalization by allowing some extrastructure over $\aleph_\alpha$, e.g. some equivalence relation; then in K instead of M_α a set $\mathcal{M}_\alpha$ of models appear, and in the definition of interpretation $\mathcal{M}_\alpha \models \psi$ means $M \in \mathcal{M}_\alpha$ implies $M \models \psi$. However, if we allow (in the case of equivalence relations) quantification over permutations, we get full second order theory. But we can allow quantification only over automorphisms of $\aleph_\alpha$ with the extrastructure. Even for a one-place function this is equivalent to a full second-order theory. Contrast this with the decidability of the monadic second-order theory of a one-place function, which is shown by Le Tourneau to follow from Rabin [13]. But for equivalence relations ordered by refinement, we get:

THEOREM 5.3. *$K^{12,n}$, $K^{13,n}$ are bi-interpretable, where*

$$\mathscr{M}^{12} = \{\langle \aleph_\alpha, \mathrm{Aut}_\alpha(\bar{e}); \bar{e}\rangle : \bar{e} = (e_1, \cdots, e_n),\ e_i \in E_\alpha^{\aleph_{\alpha+1}},\ e_i \text{ refines } e_{i+1}\},$$

$M_\alpha^{13} = \langle \alpha, U_\alpha^n; <\rangle$ where $<$ is the order of ordinals and $|U_\alpha^0| = \min(2^{\aleph_0}, \aleph_\alpha)$, $|U_\alpha^{n+1}| = \min(|\alpha|^{|U_\alpha^n|}, \aleph_\alpha)$ where $\mathrm{Aut}_\alpha(\bar{e})$ is the set of automorphisms of $(\aleph_\alpha, e)$.

We could have generalized Theorem 5.3 in the direction of Theorem 5.2, replacing (or adding to) the automorphisms by appropriate sets of equivalence relations.

For $M_\alpha^{14} = \{\langle \aleph_\alpha, \mathrm{Aut}(<); <\rangle : <$ linearly orders $\aleph_\alpha\}$ K^{14} is in fact bi-interpretable with second order logic. On the other hand, by Rabin [13], the monadic second-order theory of countable orders is decidable; and for not necessarily countable orders, the decidability is conjectured.

We can look also at $\tilde{E}_\alpha^\kappa$, i.e., the set of equivalence relations over $\aleph_\alpha$ with $< \kappa$ equivalence classes. It is not hard to see that in $\langle \aleph_\alpha, E_\alpha^\kappa; \rangle$ we can interpret $\tilde{E}_\alpha^\lambda$ when $(\Sigma_{\mu<\kappa} 2^\mu)^+ = \lambda$. However, the converse is not true. If $\aleph_\alpha \geqq \aleph_\beta \geqq \kappa$, κ regular (or $\aleph_\beta > \kappa$) then $\langle \aleph_\alpha, \tilde{E}_\alpha^\kappa; \rangle$ is an elementary extension of $\langle \aleph_\alpha^\alpha, \tilde{E}^\kappa; \rangle$. The theory of the natural numbers with addition and multiplication, and the theory of $\langle \aleph_0, \tilde{E}_0^{\aleph_0}; \rangle$ are bi-interpretable (one recursive in the other).

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